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ON EIGENVALUES AND MAIN EIGENVALUES OF A GRAPH

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Abstract. Let G be a simple graph of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ be its eigenvalues with respect to the ordinary adjacency matrix $A = A(G)$ and the Seidel adjacency matrix $A^* = A^*(G)$, respectively. Using the Courant-Weyl inequalities we prove that $\overline{\lambda}_{n+1-i} \in [-\lambda_i - 1, -\lambda_{i+1} - 1]$ and $\lambda^*_{n+1-i} \in [-2\lambda_i - 1, -2\lambda_{i+1} - 1]$ for $i = 1, 2, \ldots, n-1$, where $\overline{\lambda}_i$ are the eigenvalues of its complement $\overline{G}$. Besides, if G and H are two switching equivalent graphs then we find $\lambda_i(G) \in [\lambda_{i+1}(H), \lambda_{i-1}(H)]$ for $i = 2, 3, \ldots, n-1$. Next, let $\mu_1, \mu_2, \ldots, \mu_k$ and $\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_k$ denote the main eigenvalues of the graph G and the complementary graph $\overline{G}$, respectively. In this paper we also prove $\sum_{i=1}^{k}(\mu_i + \overline{\mu}_i) = n - k$.

1. Introduction

Let G be a simple graph of order n. The spectrum of such a graph contains the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix $A = A(G)$ and is denoted by $\sigma(G)$. The Seidel spectrum contains the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its Seidel (0,−1,1) adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P^*_G(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively.

In the sequel, for the sake of brevity, eigenvalues $\lambda_i(\overline{G})$ of the complementary graph $\overline{G}$ will be denoted by $\overline{\lambda}_i$ $(i = 1, 2, \ldots, n)$.

Let $A = A[a_{ij}]$ be the adjacency matrix of a graph G and let $A^k = A^k[a_{ij}^{(k)}]$ for any non-negative integer k. The number N_k of all walks of length k in G equals $\mathbf{sum}\, A^k$, where $\mathbf{sum}\, M$ is the sum of all elements in a square matrix M.

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As is known,

$$(1)\qquad H_G(t)=\sum_{k=0}^{\infty}N_kt^k \qquad \big(|t|<\lambda_1^{-1}\big),$$

is the generating function of the number N_k of walks of length k in the graph G. Using (1) and according to [1], we have

$$H_G(\frac{1}{\lambda})=\lambda\left[\frac{(-1)^nP_{\overline{G}}(-\lambda-1)}{P_G(\lambda)}-1\right]>0 \qquad (\lambda>\lambda_1),$$

from which we obtain $(-1)^nP_{\overline{G}}(-\lambda-1)>P_G(\lambda)>0$ for $\lambda>\lambda_1$. Hence we find that $P_{\overline{G}}(\lambda)$ has no zero in the interval $(-\infty,-\lambda_1-1)$. Therefore, $\overline{\lambda}_n\ge-\lambda_1-1$.

Proposition 4. *For any graph G of order n we have $\lambda_1+\overline{\lambda}_n+1\ge 0$.*

This simple result has motivated us to consider some other inequalities with respect to the eigenvalues λ_i of the graph G and eigenvalues $\overline{\lambda}_i$ of its complementary graph $\overline{G}$. They are based on the following statement.

Theorem 1. (The Courant-Weyl inequalities (see [1])). *Let A and B be two real symmetric matrices of order n and let $C=A+B$. Then*

$$(1^0)\qquad \lambda_{i+j+1}(C)\le\lambda_{i+1}(A)+\lambda_{j+1}(B);$$

$$(2^0)\qquad \lambda_{n-i-j}(C)\ge\lambda_{n-i}(A)+\lambda_{n-j}(B),$$

where $0\le i,\ j,\ i+j+1\le n$.

2. Some Consequences of the Ccourant-Weyl inequalities

Proposition 5. *Let G be a graph of order n. Then*

$$(2)\qquad \lambda_i+\overline{\lambda}_{n+1-i}+1\ge 0 \qquad \big(i=1,2,\dots,n\big);$$

$$(3)\qquad \lambda_{i+1}+\overline{\lambda}_{n+1-i}+1\le 0 \qquad \big(i=1,2,\dots,n-1\big).$$

Proof. Since $K=A+\overline{A}$ is the adjacency matrix of the complete graph K_n, we have that $\lambda_1(K_n)=n-1$ and $\lambda_i(K_n)=-1$ for $i=2,3,\dots,n$.

Setting $i+j+1=n$ and replacing $i+1$ with i in (1^0), we obtain (2). Similarly, setting $n-i-j=2$ and replacing $n-i$ with $i+1$ in (2^0), we obtain (3).□

D. Cvetković in [5] proved: (i) $\lambda_i+\overline{\lambda}_j+1\ge n\,\delta_{2,i+j}$ and (ii) $\lambda_{n-i+1}+\overline{\lambda}_{n-j+1}+1\le n\,\delta_{n+1,i+j}$, where $2\le i+j\le n+1$ and $\delta_{i,j}$ is the Kronecker delta symbol. Note that (2) and (3) (Proposition 2) can be easily proved by using relation (i) and (ii), respectively. For instance, replacing j with $n+1-i$ in inequality (i) we obtain implicitly Proposition 2 (2).

Corollary 1. *Let G be a graph of order n. Then*

$$\overline{\lambda}_{n+1-i} \in \big[-\lambda_i - 1,\ -\lambda_{i+1} - 1\big] \qquad \big(i = 1, 2, \dots , n-1\big).$$

The following result is well-known in the spectral theory of graphs, for instance see [1], [3].

Corollary 2. *Let $\lambda \in \sigma(G)$ be an eigenvalue of the graph G with multiplicity $p \geq 1$ and let q be the multiplicity of the eigenvalue $-\lambda - 1 \in \sigma(\overline{G})$. Then $p - 1 \leq q \leq p + 1$.*

Corollary 3. *For any graph G of order n we have $|\lambda_n| + |\overline{\lambda}_n| \leq 2\lambda_* + 1$, where $\lambda_* = \min\{\lambda_1, \overline{\lambda}_1\}$.*

Proof. Since $\lambda_1 + \overline{\lambda}_n + 1 \geq 0$ and $\lambda_1 + \lambda_n \geq 0$ we easily get $|\lambda_n| + |\overline{\lambda}_n| \leq 2\lambda_1 + 1$, which provides the proof.□

For the complete graph K_n and $G = \bigcup_{i=1}^{k} K_2$ note that $|\lambda_n| + |\overline{\lambda}_n| = 2\lambda_* + 1$ for any $k \in \mathbb{N}$.

Further, let G be a regular graph of order n and degree r. It is known that $n - 1 - r \geq -\lambda_n - 1 \geq \cdots \geq -\lambda_2 - 1$ are the eigenvalues of the regular graph $\overline{G}$. Using this fact and having in mind that $\lambda_1 + \overline{\lambda}_1 = n - 1$ if and only if G is regular, we can easy to prove the next statement.

Proposition 6. *Let G be a graph of order n. Then $\lambda_{i+1} + \overline{\lambda}_{n+1-i} + 1 = 0$ for $i = 1, 2, \dots n-1$ if and only if G is regular.*

It is also well-known that $A^* + 2A = K$. Using the same arguments as in the proof of Proposition 2, one may obtain the following result.

Proposition 7. *Let G be a graph of order n. Then:*

$$2\lambda_i + \lambda^*_{n+1-i} + 1 \geq 0 \qquad \big(i = 1, 2, \dots , n\big); \tag{4}$$

$$2\lambda_{i+1} + \lambda^*_{n+1-i} + 1 \leq 0 \qquad \big(i = 1, 2, \dots , n-1\big). \tag{5}$$

Corollary 4. *Let G be a graph of order n. Then*

$$\lambda^*_{n+1-i} \in \big[-2\lambda_i - 1,\ -2\lambda_{i+1} - 1\big] \qquad \big(i = 1, 2, \dots , n-1\big).$$

Corollary 5. ([2], [4]). *Let $\lambda \in \sigma(G)$ be an eigenvalue of the graph G with multiplicity $p \geq 1$ and let q be the multiplicity of the eigenvalue $-2\lambda - 1 \in \sigma^*(G)$. Then $p - 1 \leq q \leq p + 1$.*

Let S be any subset of the vertex set $V(G)$. To switch G with respect to S means to remove all edges connecting S with $\overline{S} = V(G) \setminus S$, and to introduce an edge between all nonadjacent vertices in G which connect S with $\overline{S}$. Two graphs G and H are switching equivalent if one of them is obtained from the other by switching. It is known that switching equivalent graphs have the same Seidel spectrum.

Proposition 8. *Let G and H be two switching equivalent graphs. Then we have $\lambda_i(G) \geq \lambda_{i+1}(H)$ for $i = 1, 2, \ldots, n-1$.*

Proof. Replacing $i+1$ with i in (5) and using (4), we immediately obtain

$$\lambda_i \in \left[-\frac{\lambda^*_{n+1-i}+1}{2}, -\frac{\lambda^*_{n+2-i}+1}{2}\right] \qquad \left(i = 2, 3, \ldots, n\right).$$

Using that $\sigma^*(G) = \sigma^*(H)$ and according to (4) and the last relation, we get easily

$$\lambda_1(G),\ \lambda_1(H) \geq \lambda_2(G),\ \lambda_2(H) \geq \cdots \geq \lambda_n(G),\ \lambda_n(H),$$

from which we obtain the statement. □

Due to the fact that $\lambda \in \sigma(G)$ cannot be a non-integer rational number, we obtain a result as follows.

Corollary 6. *If $\lambda^* \in \sigma^*(G)$ is an even integer then its multiplicity must be* 1.

Corollary 7. *Let G and H be two switching equivalent graphs. Then*

$$\lambda_i(G) \in \left[\lambda_{i+1}(H),\ \lambda_{i-1}(H)\right] \qquad \left(i = 2, 3, \ldots, n-1\right).$$

Corollary 8. ([4]). *Let G and H be two switching equivalent graphs. Let $\lambda \in \sigma(G)$ be an eigenvalue of the graph G with multiplicity $p \geq 1$ and let q be the multiplicity of λ with respect to H. Then $p-2 \leq q \leq p+2$.*

For a graph G let $n_+(G)$ and $n_-(G)$ denote the number of positive and negative eigenvalues of G, respectively.

Corollary 9. *Let G and H be two switching equivalent graphs. Then*

$$n_+(G) - 1 \leq n_+(H) \leq n_+(G) + 1;$$
$$n_-(G) - 1 \leq n_-(H) \leq n_-(G) + 1.$$

Definition 1. *A graph G of order n is called spectral complementary, if*

$$P_G(\lambda) - P_{\overline{G}}(\lambda) = (-1)^n \left(P_G(-\lambda-1) - P_{\overline{G}}(-\lambda-1)\right).$$

Some elementary results related to the spectral complementary graphs were proved in [7]. Among other results, it was proved the following:

(x) $G \cup \overline{G}$ is spectral complementary for any graph G;

(y) G is spectral complementary if and only if $\sigma^*(G) = \sigma^*(\overline{G})$;

(z) there exists no spectral complementary graph of order $4n+3$ for any $n \geq 0$.

According to (x), one can see that the class of all spectral complementary graphs is infinite. We notice, according to (z), the class of all graphs which are not spectral complementary is infinite too.

Proposition 9. *Let G be a spectral complementary graph. Then*

$$\lambda_i \geq \overline{\lambda}_{i+1} \quad \text{and} \quad \overline{\lambda}_i \geq \lambda_{i+1},$$

for $i = 1, 2, \ldots, n-1$.

Proof. Using (y) and (5), we get

$$2\overline{\lambda}_{i+1} + \lambda^*_{n+1-i} + 1 \leq 0 \qquad (i = 1, 2, \ldots, n-1).$$

Combining (4), (y) and the last relation, we obtain the statement.□

Corollary 10. *Let G be a spectral complementary graph. Then*

$$n_+(G) - 1 \leq n_+(\overline{G}) \leq n_+(G) + 1;$$
$$n_-(G) - 1 \leq n_-(\overline{G}) \leq n_-(G) + 1.$$

For any $n \geq 3$ the complete graph K_n violates some relations in Proposition 6 and/or Corollary 10. Thus, K_n is not spectral complementary for any $n \geq 3$.

On the other hand, the unicyclic graph C_4 is consistent with all relations in Proposition 6 and/or Corollary 10 – in spite of this fact, C_4 is not spectral complementary.

3. Some results on main eigenvalues

Let $\mu_1 > \mu_2 > \cdots > \mu_m$ be the distinct eigenvalues of a graph G of order n and let $\mathcal{E}_A(\mu_i)$ denote the eigenspace of the eigenvalue μ_i $(i = 1, 2, \ldots, m)$.

An eigenvalue $\mu \in \sigma(G)$ is called the main eigenvalue if the cosine of the angle between the eigenspace $\mathcal{E}_A(\mu)$ and the main vector $\mathbf{j}$ (whose all coordinates are equal to 1) is different from zero. In other words, we say that the eigenvalue μ is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j} \rangle = n \cos^2 \beta \neq 0$, where $\mathbf{P}$ represents the orthogonal projection of the space $\mathbb{R}^n$ onto $\mathcal{E}_A(\mu)$.

Let $\mathcal{M}(G)$ denote the set of all main eigenvalues of a graph G. As is known, G and its complementary graph $\overline{G}$ have the same number of main eigenvalues [3], that is $|\mathcal{M}(G)| = |\mathcal{M}(\overline{\mathcal{G}})|$.

The class of all spectral complementary graphs is very large. For the sake of an example, between 11 117 of all connected non-isomorphic graphs of order 8 there exist exactly 1142 spectral complementary graphs (see [7]).

Proposition 10. *Let $\mu_1, \mu_2, \ldots, \mu_k$ and $\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_k$ be the main eigenvalues of the graph G and its complement $\overline{G}$, respectively. Then we have* $\sum\limits_{i=1}^{k} \left(\mu_i + \overline{\mu}_i\right) = n - k$.

Proof. Since

$$\sum_{i=1}^{k} \mu_i + \sum_{i \in \Lambda} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{k} \overline{\mu}_k + \sum_{i \in \Lambda} (-\lambda_i - 1) = 0\,,$$

where $\Lambda = \{\, i \mid \lambda_i \in \sigma(G) \setminus \mathcal{M}(G)\}$, we readily obtain the proof.□

Corollary 11. *Let G be a self-complementary graph of order n. Then*

$$n \equiv k \,(mod\ 2) \quad where \quad k = |\mathcal{M}(G)|\,.$$

Proof. Setting $\Delta(G) = \sigma(G) \setminus \mathcal{M}(G)$ note that $\Delta(G) = \Delta(\overline{G})$. Consequently, for any $\lambda \in \Delta(G)$ we have $-\lambda - 1 \in \Delta(G)$. Since $\lambda = -\lambda - 1$ is impossible for any λ, $-\lambda - 1 \in \Delta(G)$, it turns out that $|\Delta(G)| = n - k$ must be an even integer.□

Corollary 12. *There exists no regular self-complementary graph of order $2n$.*

Let $\mu_1^* > \mu_2^* > \cdots > \mu_m^*$ be the distinct Seidel eigenvalues of a graph G of order n. In accordance with [4], $\lambda^* \in \sigma^*(G)$ is called the Seidel main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^*\mathbf{j} \rangle = n \cos^2 \beta^* \neq 0$, where $\mathbf{P}^*$ is the orthogonal projection of the space $\mathbb{R}^n$ onto $\mathcal{E}_{A^*}(\lambda^*)$. The number $\cos \beta^*$ is called the Seidel main angle of λ^*. Let $\mathcal{M}^*(G)$ denote the set of all Seidel main eigenvalues of a graph G. Then we also have $|\mathcal{M}(G)| = |\mathcal{M}^*(G)|$ (see [4]).

Corollary 13. *Let $\lambda^* \in \sigma^*(G)$ be an even integer. Then $\lambda^* \in \mathcal{M}^*(G)$.*

Proof. If $\lambda^* \in \sigma^*(G)$ is an even integer then $-\dfrac{\lambda^* + 1}{2} \notin \sigma(G)$, which provides the proof. □

Due to the fact that the Seidel spectrum of spectral complementary graphs is symmetric with respect to the zero point, we arrive at

Corollary 14. *Let G be a spectral complementary graph of order $2n$. Then $0 \notin \sigma^*(G)$*

Proof. Let assume, contrary to the statement, that $0 \in \sigma^*(G)$. Then its multiplicity must be a positive even number, which is a contradiction to Corollary 6.

This completes the proof.□

Proposition 11. *Let G be a regular graph of order n such that* $P_G(\lambda) = P_{\overline{G}}(\lambda)$. *Then* $0 \in \mathcal{M}^*(G) \cap \mathcal{M}^*(\overline{G})$.

Proof. Since $P_G(\lambda) = P_{\overline{G}}(\lambda)$ it is clear that G is a spectral complementary graph (see Definition 1). Consequently, using (y) we have $P^*_G(\lambda) = P^*_{\overline{G}}(\lambda)$. Next, by relation $\lambda_1 + \overline{\lambda}_1 = n - 1$ it follows that n must be an odd integer. Thus, $0 \in \sigma^*(G)$.

This provides the proof.□

Using that $A^* + 2A = K$ and following the same procedure as in the proof of Proposition 7, we obtain

Corollary 15. *Let* $\mu_1, \mu_2, \ldots, \mu_k$ *and* $\mu_1^*, \mu_2^*, \ldots, \mu_k^*$ *be the main eigenvalues and the Seidel main eigenvalues of a graph G of order n, respectively. Then we have* $\sum_{i=1}^{k} (\mu_i^* + 2\mu_i) = n - k$.

Further, for any $\lambda^* \in \sigma^*(G)$ we have that $-\lambda^* \in \sigma^*(\overline{G})$. Having in mind that $\mathcal{E}_{A^*}(\lambda^*) = \mathcal{E}_{\overline{A}^*}(-\lambda^*)$, we obtain implicitly $\mathcal{M}^*(\overline{G}) = -\mathcal{M}^*(G)$, understanding that $-\mathcal{M}^*(G) = \{\lambda^* \mid -\lambda^* \in \mathcal{M}^*(G)\}$.

Corollary 16. *Let G be a self-complementary graph. Then the Seidel main spectrum* $\mathcal{M}^*(G)$ *is symmetric with respect to the zero point.*

In this paper,

$$H^*_G(t) = \sum_{k=0}^{\infty} N^*_k t^k \quad \text{where} \quad N^*_k = \textbf{sum}\left(A^*\right)^k,$$

is called the Seidel generating function (see also [6]).

Using relation $2A = J - I - A^*$ and according to [1], we can easy find the next two results.

Corollary 17. *Let G be a graph of order n. Then*

$$H^*_G\left(\frac{1}{\lambda}\right) = \lambda \left[\frac{(-1)^n 2^n P_G\left(-\dfrac{\lambda+1}{2}\right)}{P^*_G(\lambda)} - 1 \right].$$

Corollary 18. *For any graph G we have*

$$H^*_G\left(-\frac{1}{2\lambda+1}\right) = \frac{(2\lambda+1)\, H_G\left(\frac{1}{\lambda}\right)}{2\lambda + H_G\left(\frac{1}{\lambda}\right)}.$$

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